Parabolic Anderson model with rough noise in space and rough initial conditions

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Jointwork with

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Stochastic Heat Equation / Parabolic Anderson Model

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \dfrac{1}{2}\Delta\right) u(t,x) = u(t,x)\dot{W}(t,x), & t > 0,\ x \in \mathbb{R} \\ u(0,\cdot) = \mu_0 \end{cases}$$

1. $\dot{W}$: Centered Gaussian noise that is homogeneous in space;
2. μ_0: Initial (nonnegative) measure.

$$u(t,x) = J_0(t,x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)u(s,y)W(\mathrm{d}s,\mathrm{d}y), \qquad \text{(Skorohod)}$$

where J_0 is the solution to the homogeneous heat equation, i.e.,

$$J_0(t,x) := \int_{\mathbb{R}} p_t(x-y)\mu_0(\mathrm{d}y) \quad \text{with} \quad p_t(x) = (2\pi t)^{-1/2} e^{-\frac{x^2}{2t}}.$$

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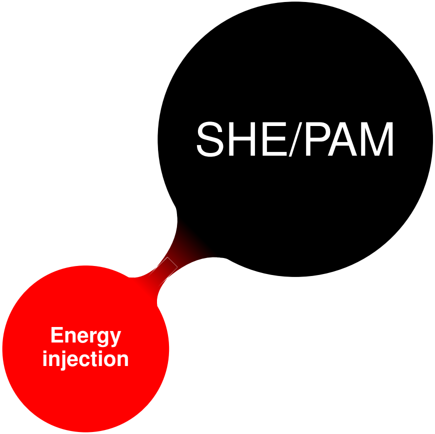
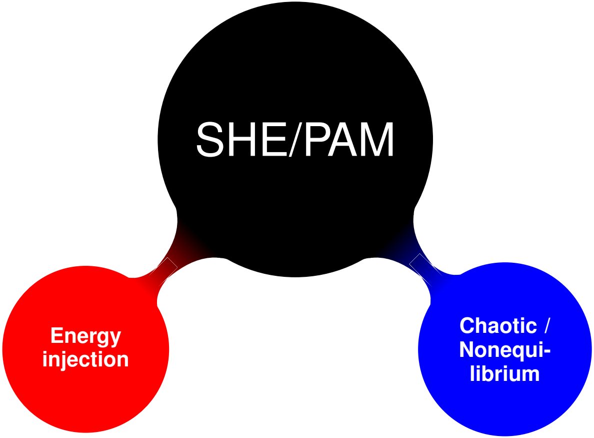
SHE/PAM


SHE/PAM
Energy injection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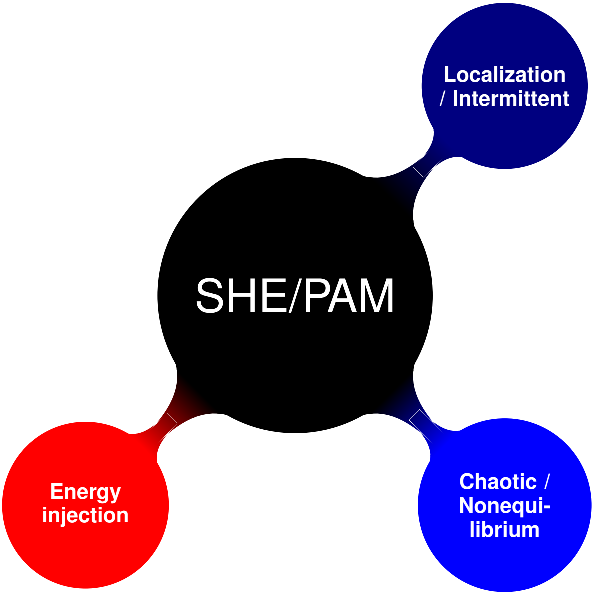
Localization / Intermittent
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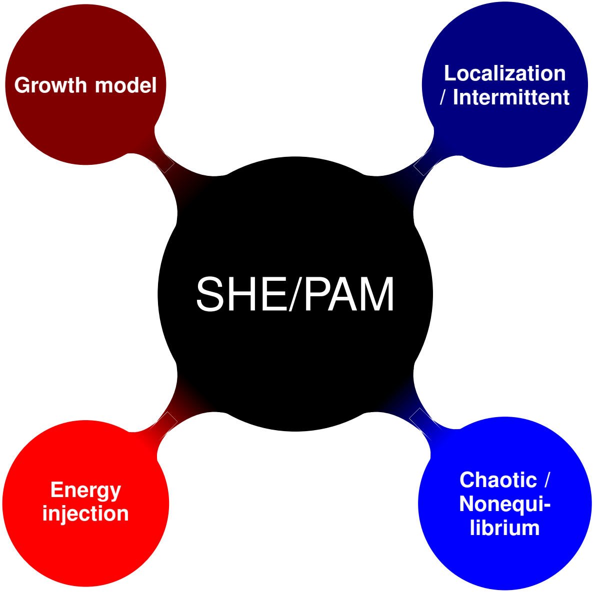
Growth model
Localization / Intermittent
SHE/PAM
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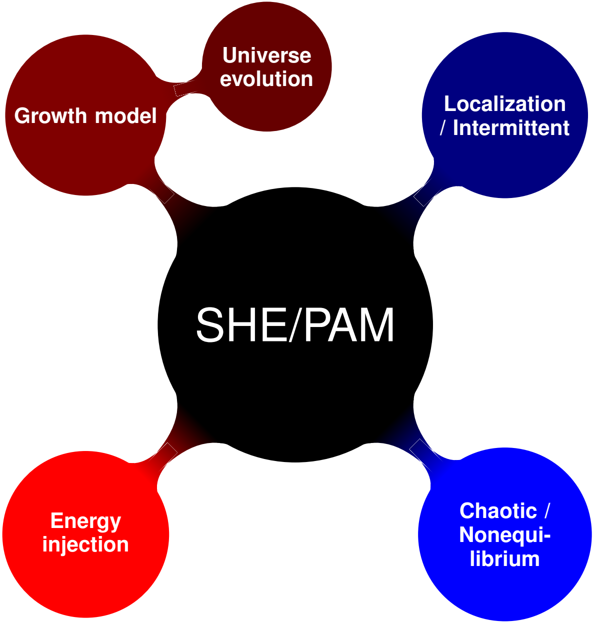
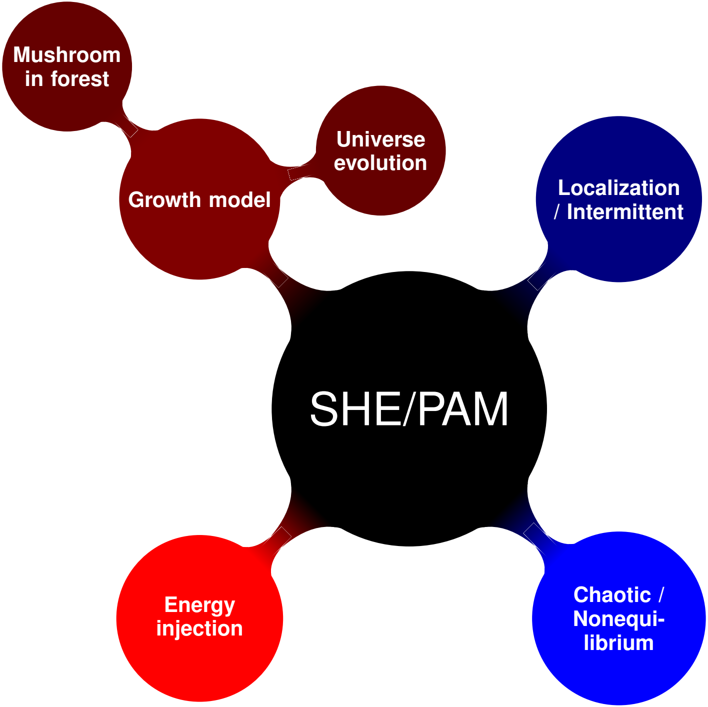
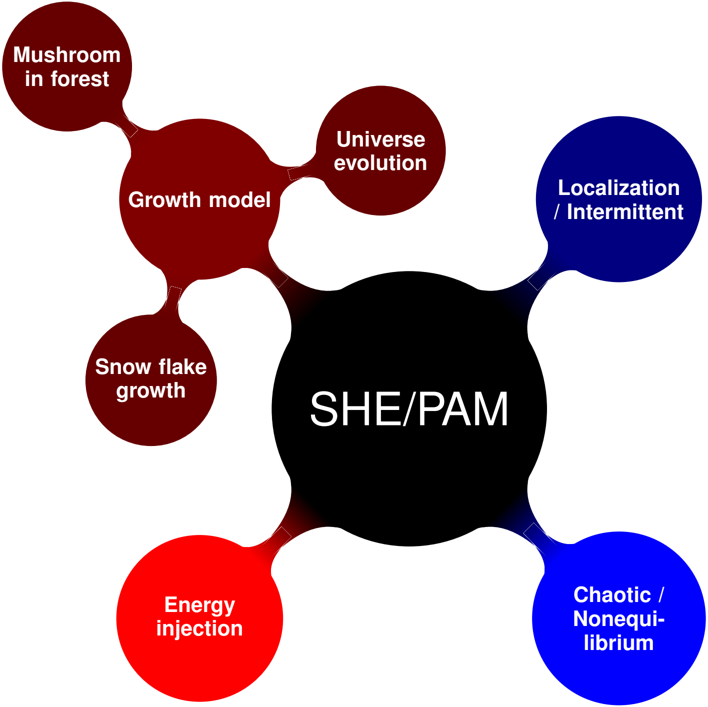

SHE/PAM
Growth model
Universe evolution
Localization / Intermittent
Energy injection
Chaotic / Nonequi- librium


Mushroom in forest
Universe evolution
Growth model
Localization / Intermittent
SHE/PAM
Energy injection
Chaotic / Nonequi-librium


Mushroom in forest
Universe evolution
Growth model
Localization / Intermittent
Snow flake growth
SHE/PAM
Energy injection
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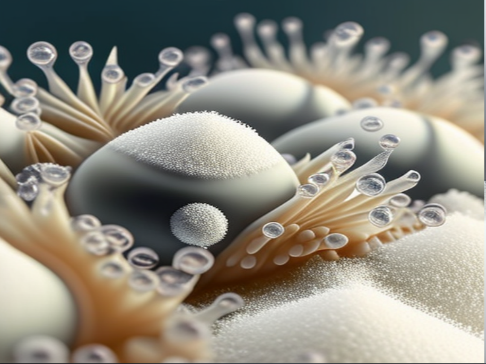



THE ELLVSSTIUN


SCO
1SVE


Centered Gaussian noise

(homogeneous in space)

Homogeneous
in space

$$\mathbb{E}\left(\dot{W}(t,x)\dot{W}(s,y)\right) = C_{H_0,H}|t-s|^{2H_0-2}\,|x-y|^{2H-2}$$

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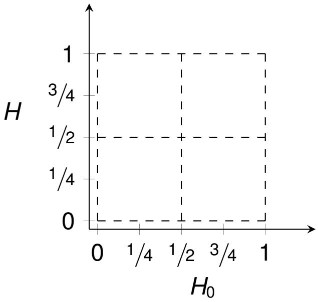

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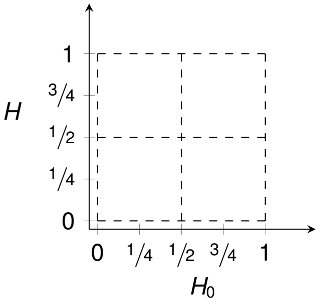

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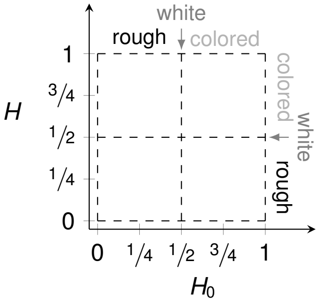

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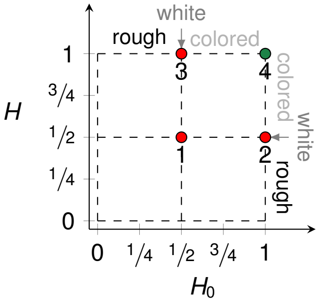

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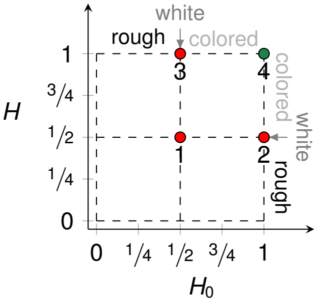

1. space-time white noise
2. time-indep. space white noise
3. space-indep. time white noise
4. deterministic ponential

Centered Gaussian noise

(homogeneous in space)

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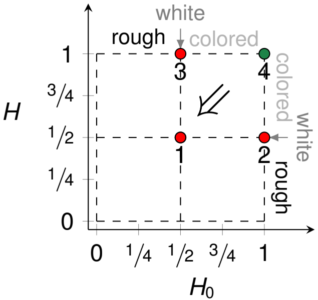

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$\partial_t - \frac{1}{2}\Delta$ $\qquad$ $u(t,x)\dot{W}(t,x)$ $\qquad$ μ_0

$$u(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\,u(s,y)\,W(\mathrm{d}s,\mathrm{d}y)$$

$$+\left(p_t * \mu_0\right)(x)$$

$\partial_t - \frac{1}{2}\Delta$ $u(t,x)\dot{W}(t,x)$ μ_0

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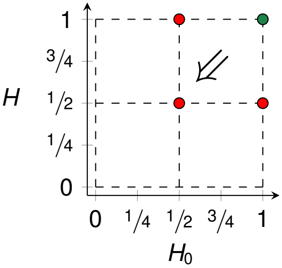

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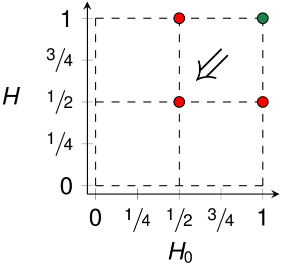

$C_c^\infty(\mathbb{R})$

1 $\qquad$ $|x|^{-1/2}$

$|x|^2$ $\qquad$ δ_0

$e^{|x|^{3/2}}$ $\qquad$ $\Downarrow$ $\qquad$ $|x|^{-3/2}$

$e^{|x|^3}$ $\qquad$ δ_0'

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Rough Initial Data

$e^{|x|^{3/2}}$ $|x|^{-3/2}$

$e^{|x|^3}$ δ_0'

$$(p_t * \mu_0)(x) < \infty \text{ for all } t > 0 \text{ and } x \in \mathbb{R}$$

$$\Updownarrow$$

RID: $\displaystyle\int_{\mathbb{R}} e^{-a|x|^2} \mu_0(\mathrm{d}x) < \infty$ for all $a > 0$

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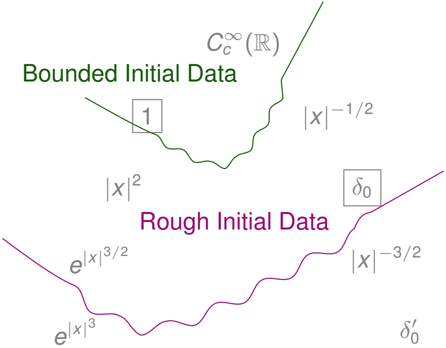
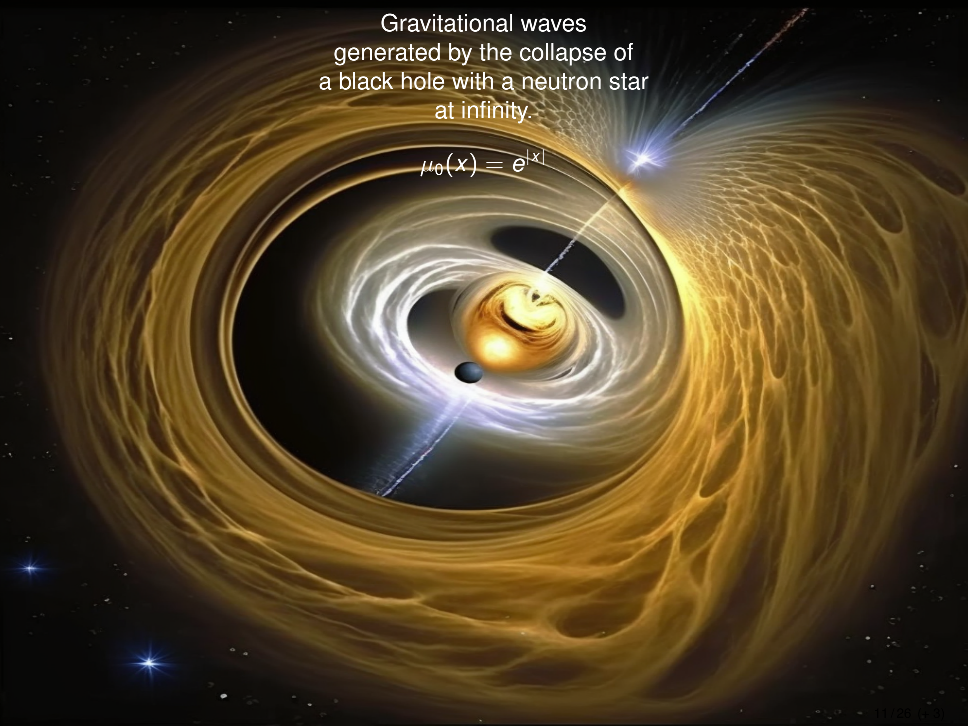


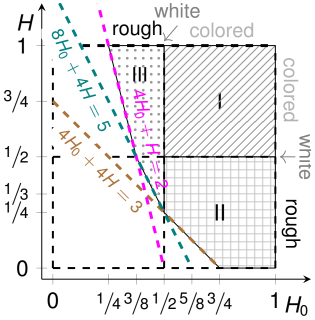
First sun light in the morning ($\mu_0(x) = 1$)

Explosion ($\mu_0 = \delta_0$)

Gravitational waves
generated by the collapse of
a black hole with a neutron star
at infinity.

$$\mu_0(x) = e^{|x|}$$



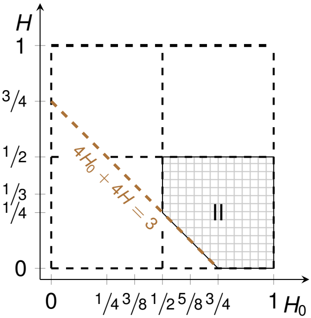


Region II	Initial data	Moments
Hu et al., 2018	BIC	Matching bds
Hu and Lê, 2019	BIC–RIC †	Upper bds
X. Chen, 2019	BIC	Exact asymptotics
Z.-Q. Chen and Hu, 2021	BIC (Necessary)	—
R. Balan et al., 2022 (L.C.)	RIC	Upper bds

Theorem (R. Balan et al., 2022 (L.C.)) If (H_0, H) fall in Region II, and if μ_0 is a rough initial condition, then there is a unique solution u which satisfies:

$$\mathbb{E}\left(|u(t,x)|^p\right) \le C_1^p J_0^p(t,x) \exp\left(C_2 p^{\frac{H+1}{H}} t^{\frac{2H_0+H-1}{H}}\right), \quad \forall (t,x,p) \in \mathbb{R}_+ \times \mathbb{R} \times [2,\infty),$$

where $C_1 > 0$ and $C_2 > 0$ are some constants which depend on H_0 and H.



$$4H_0 + 4H > 3$$

$$\Downarrow$$

$$\frac{2H_0 + H - 1}{H} \ge \frac{1}{2H} - 1 > 0$$

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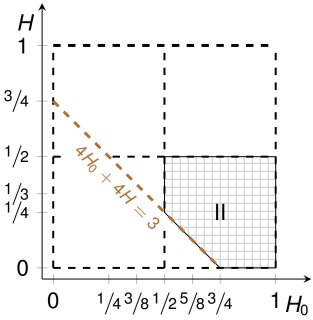

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BIC–RIC † (Hu and Lê, 2019): For some $C > 0$ and $\beta < H_0$,

$$\int_{\mathbb{R}} \left(1 + |\xi|^{-(H-1/2)}\right) e^{-t|\xi|^2} |\mathcal{F}\mu_0(\xi)| \mathrm{d}\xi \leq C t^{-\beta} \quad \text{for all } t > 0.$$

RIC: $\displaystyle\int_{\mathbb{R}} e^{-ax^2} \mu_0(\mathrm{d}x) < \infty, \quad \text{for all } a > 0.$

$\mu_0(x)$	$\mathcal{F}(\mu_0)(\xi)$	$\lvert\mathcal{F}(\mu_0)(\xi)\rvert$	BIC	BIC–RIC †	RIC
1	$\delta_0(\xi)$	$\delta_0(\xi)$	✓	✓	✓
$\lvert x\rvert^{-1/3}$	$\lvert\xi\rvert^{-2/3}$	$\lvert\xi\rvert^{-2/3}$	✗	✓	✓
$\delta_0(x)$	1	1	✗	✓	✓
x^2	$\delta_0''(\xi)$	—	✗	✗	✓
$e^{\lvert x\rvert}$	—	—	✗	✗	✓

Chaos expansion:

$$u(t,x) = J_0(t,x) + \sum_{n\geq 1} I_n(f_n(\cdot,t,x))$$

with

$$f_n(t_1,x_1,\ldots,t_n,x_n,t,x) := \prod_{j=1}^{n} p_{t_{j+1}-t_j}(x_{j+1}-x_j) J_0(t_1,x_1) 1_{\{0<t_1<\ldots<t_n<t\}}.$$

symmetrization:

$$\widetilde{f}_n(t_1,x_1,\ldots,t_n,x_n,t,x) = \frac{1}{n!}\sum_{\rho\in S_n} f_n(t_{\rho(1)},x_{\rho(1)},\ldots,t_{\rho(n)},x_{\rho(n)},t,x).$$

Isometry:

$$
\begin{aligned}
||u(t,x)||_2^2 &= \sum_{n\geq 1} n! \|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} < \infty \\
&= \sum_{n\geq 1} n! \iint_{[0,t]^{2n}} \mathrm{d}\vec{t}\,\mathrm{d}\vec{s} \int_{\mathbb{R}^d} \mu(\mathrm{d}\vec{\xi}) \left(\prod_{j=1}^n |t_j - s_j|^{2H_0-2}\right) \\
&\qquad \times \mathcal{F}\widetilde{f}_n(t_1,\cdot,\dots,t_n,\cdot,t,x)(\xi_1,\dots,\xi_n) \\
&\qquad \times \overline{\mathcal{F}\widetilde{f}_n(s_1,\cdot,\dots,s_n,\cdot,t,x)(\xi_1,\dots,\xi_n)}.
\end{aligned}
$$

Isometry:

$$\|u(t,x)\|_2^2 = \sum_{n\geq 1} n! \|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} < \infty$$

$$= \sum_{n\geq 1} n! \iint_{[0,t]^{2n}} \mathrm{d}\vec{t}\, \mathrm{d}\vec{s} \int_{\mathbb{R}^d} \mu(\mathrm{d}\vec{\xi}) \left(\prod_{j=1}^{n} |t_j - s_j|^{2H_0-2} \right)$$

$$\times \mathcal{F}\widetilde{f}_n(t_1,\cdot,\ldots,t_n,\cdot,t,x)(\xi_1,\ldots,\xi_n)$$

$$\times \overline{\mathcal{F}\widetilde{f}_n(s_1,\cdot,\ldots,s_n,\cdot,t,x)(\xi_1,\ldots,\xi_n)}.$$

Integrate $\mathrm{d}\vec{t}\,\mathrm{d}\vec{s}$ first? Integrate $\mu(\mathrm{d}\vec{\xi})$ first?


“Two roads diverged ... ...,
:
I took the one less traveled by,
And that has made all differences.”
—Robert Frost

X. Chen, 2019 integrate $\mathrm{d}\vec{t}\,\mathrm{d}\vec{s}$ first:

- Double exponential trick
- Laplace method

We has to integrate $\mu(\mathrm{d}\vec{\xi})$ first:

- Laplace transform doesn't apply
- Density for Brownian bridge nonlinear in t

Set

$$\psi_{t,x}^{(n)}(\vec{t},\vec{s}) := (n!)^2 \int_{\mathbb{R}^d} \mu(\mathrm{d}\vec{\xi})\mathcal{F}\widetilde{f}_n(t_1,\cdot,\ldots,t_n,\cdot,t,x)(\xi_1,\ldots,\xi_n) \times \overline{\mathcal{F}\widetilde{f}_n(s_1,\cdot,\ldots,s_n,\cdot,t,x)(\xi_1,\ldots,\xi_n)}.$$

so that

$$||u(t,x)||_2^2 = \sum_{n\geq 1}\frac{1}{n!}\iint_{[0,t]^{2n}} \mathrm{d}\vec{t}\mathrm{d}\vec{s}\left(\prod_{j=1}^n |t_j-s_j|^{2H_0-2}\right)\psi_{t,x}^{(n)}\left(\vec{t},\vec{s}\right)$$
$$\leq b_{H_0}^n \sum_{n\geq 1}\frac{1}{n!}\left(\int_{[0,t]^n}\mathrm{d}\vec{t}\left|\psi_{t,x}^{(n)}\left(\vec{t},\vec{t}\right)\right|^{1/H_0}\right)^{2H_0}.$$

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&\leq b_{H_0}^{n} \sum_{n\geq 1} \frac{1}{n!} \left(\int_{[0,t]^n} \mathrm{d}\vec{t} \left| \psi_{t,x}^{(n)}\left(\vec{t},\vec{t}\right) \right|^{1/H_0} \right)^{2H_0}.
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$$||u(t,x)||_2^2 = \sum_{n\geq 1}\frac{1}{n!}\iint_{[0,t]^{2n}} \mathrm{d}\vec{t}\mathrm{d}\vec{s}\left(\prod_{j=1}^n |t_j - s_j|^{2H_0-2}\right)\psi_{t,x}^{(n)}\left(\vec{t},\vec{s}\right)$$

$$\leq b_{H_0}^n \sum_{n\geq 1}\frac{1}{n!}\left(\int_{[0,t]^n}\mathrm{d}\vec{t}\left|\psi_{t,x}^{(n)}\left(\vec{t},\vec{t}\right)\right|^{1/H_0}\right)^{2H_0}.$$

X. Chen, 2019 integrate $\mathrm{d}\vec{t}\,\mathrm{d}\vec{s}$ first:

- Double exponential trick
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$$\begin{aligned} \|u(t,x)\|_2^2 &= \sum_{n\geq 1} \frac{1}{n!} \iint_{[0,t]^{2n}} \mathrm{d}\vec{t}\mathrm{d}\vec{s} \left(\prod_{j=1}^{n} |t_j - s_j|^{2H_0-2} \right) \psi_{t,x}^{(n)}\left(\vec{t},\vec{s}\right) \\ &\leq b_{H_0}^n \sum_{n\geq 1} \frac{1}{n!} \left(\int_{[0,t]^n} \mathrm{d}\vec{t} \left| \psi_{t,x}^{(n)}\left(\vec{t},\vec{t}\right) \right|^{1/H_0} \right)^{2H_0}. \end{aligned}$$

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If $0 < t_{\rho(1)} < \ldots < t_{\rho(n)} < t =: t_{\rho(n+1)}$, then

$$\psi_{t,x}^{(n)}(\vec{t},\vec{t}) \le J_0^2(t,x) \underbrace{\int_{\mathbb{R}^n} \mu(\mathrm{d}\vec{\xi}) \prod_{k=1}^{n} \exp\left\{ -\frac{t_{\rho(k+1)} - t_{\rho(k)}}{t_{\rho(k+1)}\, t_{\rho(k)}} \left| \sum_{j=1}^{k} t_{\rho(j)} \xi_j \right|^2 \right\}}_{=:\, I_t^{(n)}(\vec{t}\,)}.$$

(Ref. Lemma 3.2 of R. M. Balan and Chen, 2018)

$\mu(\mathrm{d}\vec{\xi})$ **integral**:

1. Change-of-variable
2. Triangle inequality + Subadditivity:

$$|\eta_k - \eta_{k-1}|^{1-2H} \leq (|\eta_k| + |\eta_{k-1}|)^{1-2H}$$
$$\leq |\eta_k|^{1-2H} + |\eta_{k-1}|^{1-2H}. \qquad \text{(Recall } 1-2H>0\text{)}$$

3. $$\int_{\mathbb{R}} e^{-t|\eta|^2} |\eta|^\alpha \mathrm{d}\eta = \Gamma\left(\frac{1+\alpha}{2}\right) t^{-\frac{1+\alpha}{2}}. \qquad (\forall t>0,\ \alpha>-1)$$

Triangle inequality in Step 2 above introduces the following multipliers:

$$S_n := x_1 \prod_{k=2}^{n} (x_k + x_{k-1}) = \sum_{a\in A_n} \prod_{j=1}^{n} x_j^{a_j},$$

where A_n is a set of all possible indices $a = (a_1, \ldots, a_n)$ .....

For example,

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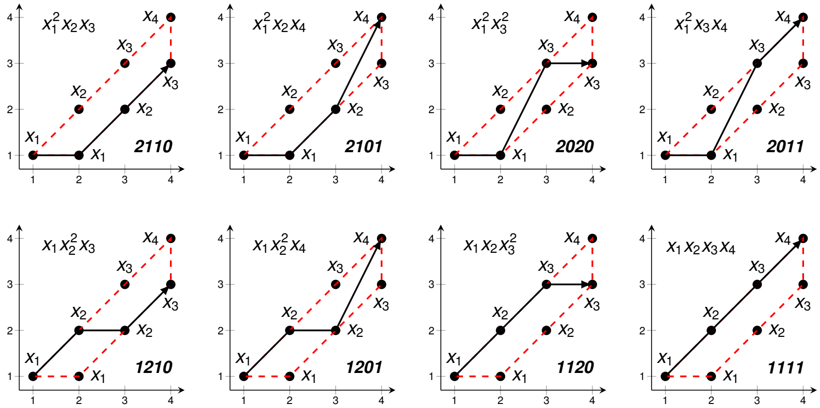

$$(a_1, \cdots, a_n) \quad \leftrightarrow \quad \text{a path from } (1,1) \text{ to either } (n,n) \text{ or } (n, n-1) \text{ within the envelope.}$$

The $\mathrm{d}\vec{t}$ integral leads to the study of the following factor:

$$\gamma_n(a_1, \ldots, a_n) = \prod_{k=1}^{n-1} \frac{\Gamma\left(\theta_k + \frac{1-2H}{4H_0}\left(a_k + a_{k+1} - 2\right)\right)}{\Gamma\left(\theta_k\right)}$$

where

$$\theta_k = \begin{cases} \dfrac{H-1}{H_0} + 2 & \text{if } k = 1, \\ 1 - \dfrac{1}{4H_0} + k\dfrac{4H_0 + 4H - 3}{4H_0} + \dfrac{1-2H}{4H_0}\displaystyle\sum_{i=1}^{k-1} a_i & \text{if } k = 2, \ldots, n, \end{cases}$$

and $(a_1, \cdots, a_n) \in A_n$.

To obtain the right exponent, one needs to establish the following lemma:

Key Lemma. $\gamma_n(a_1, \ldots, a_n) \leq \gamma_n(1, \cdots, 1) = 1$.

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Proof. Notice that $(1, \cdots, 1)$ is the up most path in the envelope. It suffices to prove that

γ_n ("path") decreases its value when the path moving downwards.

Take $(a_1, \cdots, a_n)$ and $(a'_1, \cdots, a'_n) \in A_n$ so that only the two path differs only at one location:

$$\frac{\gamma_n(a'_1, \cdots, a'_n)}{\gamma_n(a_1, \cdots, a_n)} = \prod_{k=1}^{n-1} \frac{\Gamma(\theta'_k + \frac{1-2H}{4H_0}(a'_k + a'_{k+1} - 2))}{\Gamma(\theta_k + \frac{1-2H}{4H_0}(a_k + a_{k+1} - 2))} \times \prod_{k=1}^{n-1} \frac{\Gamma(\theta_k)}{\Gamma(\theta'_k)}$$
$$= \frac{\Gamma\left(\theta_{i-1} + \frac{1-2H}{4H_0}(a_{i-1} + a_i - 2) + \frac{1-2H}{4H_0}\right)}{\Gamma\left(\theta_{i-1} + \frac{1-2H}{4H_0}(a_{i-1} + a_i - 2)\right)} \times \frac{\Gamma(\theta_{i+1})}{\Gamma\left(\theta_{i+1} + \frac{1-2H}{4H_0}\right)}.$$

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$$\frac{\gamma_n\,(a_1', \cdots, a_n')}{\gamma_n\,(a_1, \cdots, a_n)} = \prod_{k=1}^{n-1} \frac{\Gamma(\theta_k' + \frac{1-2H}{4H_0}(a_k' + a_{k+1}' - 2))}{\Gamma(\theta_k + \frac{1-2H}{4H_0}(a_k + a_{k+1} - 2))} \times \prod_{k=1}^{n-1} \frac{\Gamma(\theta_k)}{\Gamma(\theta_k')}$$
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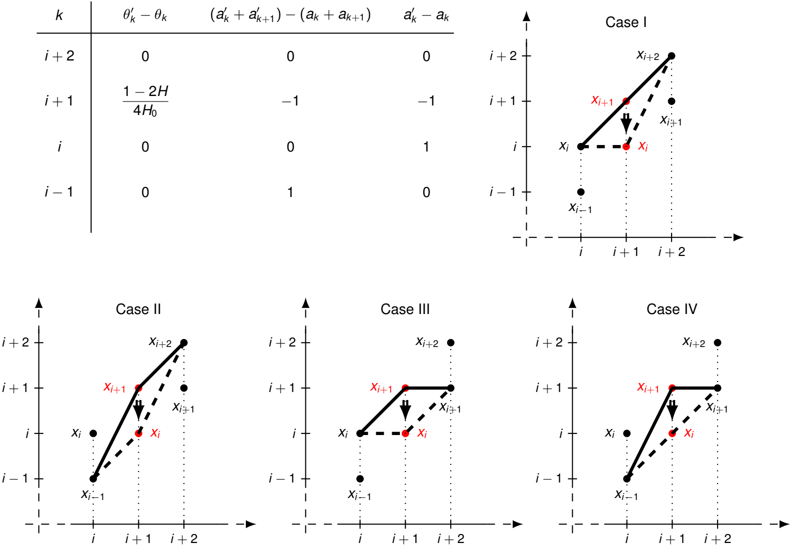
&= \frac{\Gamma\left(\theta_{i-1} + \frac{1-2H}{4H_0}(a_{i-1} + a_i - 2) + \frac{1-2H}{4H_0}\right)}{\Gamma\left(\theta_{i-1} + \frac{1-2H}{4H_0}(a_{i-1} + a_i - 2)\right)} \times \frac{\Gamma(\theta_{i+1})}{\Gamma\left(\theta_{i+1} + \frac{1-2H}{4H_0}\right)}\,.
\end{aligned}
$$

k	$\theta'_k - \theta_k$	$(a'_k + a'_{k+1}) - (a_k + a_{k+1})$	$a'_k - a_k$
$i+2$	0	0	0
$i+1$	$\frac{1-2H}{4H_0}$	-1	-1
i	0	0	1
$i-1$	0	1	0



□

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* References are produced from **SPDEs-Bib**: https://github.com/chenle02/SPDEs-Bib

* Download the bib file: https://github.com/chenle02/SPDEs-Bib/blob/main/All.bib

* Sources: **MathSciNet** and **arXiv**.

Haiku for SHE/PAM

by OpenAI's GPT-3.5

(Accessed on 2023/03/09)

Parabolic dance,
Moment asymptotics trance,
Stochastic romance.

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ChatGPT (for slides typing and haiku)

Midjourney (for generating all images)

Thank you for your listening !


Initial data is the genome
for the growth model